Analysis I Lecture 19

Next week:

- On Monday Lecture is only online
- On Wednesday. Zhipeng Xue will organize open question and Q and A on Mock exam.

Last time:

Introduced derivatives:

- f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and finite.

Then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

f is differentiable on E if it is differentiable for every $x_0 \in E$.

Computing derivatives:

Let $f, g : E \longrightarrow \mathbb{R}$ be differentiable at x_0

then

1) $f+g$ is differentiable at x_0 and

$$\boxed{(f+g)'(x_0) = f'(x_0) + g'(x_0)}$$

More generally
$(\alpha f + \beta g)' = \alpha f' + \beta g'$

2) $f \cdot g$ is differentiable at x_0 and

Leibnitz rule

$$\boxed{(f \cdot g)'(x_0) = f(x_0) \cdot g'(x_0) + f'(x_0) \cdot g(x_0)}$$

3) if $g(x_0) \neq 0$, f/g is differentiable at x_0 and

$$\boxed{\left(\frac{f}{g}\right)'(x_0) = \left(\frac{g f' - f g'}{g^2}\right)(x_0)}$$

In particular,

$$\boxed{\left(\frac{1}{g}\right)'(x_0) = \frac{-g'}{g^2}(x_0)}$$

New

Change of variables Prop. 2.18

4) $f: E \to \mathbb{R}$, $g: G \to \mathbb{R}$ s.t. $f(E) \subset G$

Assume f is differentiable at $x_0 \in E$

and g is differentiable at $f(x_0)$

then $g \circ f$ is differentiable at x_0

and

$$\boxed{(g \circ f)'(x_0) = g'\big(f(x_0)\big) \cdot f'(x_0)}$$

Chain rule

Example

$$f(x) = \tan(x^2) + 2\sin(x)\cdot\cos(x)$$

Note that $\cdot\ (x^2)' = 2x \qquad \cdot\ \sin(x)' = \cos(x)$

$\cdot\ \cos(x)' = -\sin(x)$

And $f(x)$ is obteined by arithmetic operations and composition from

$$x^2,\ \sin(x),\ \cos(x),$$

$$\left(\tan(x^2) + 2\sin(x)\cdot\cos(x)\right)' =$$

$$= \underbrace{\left(\tan(x^2)\right)'} + 2\underbrace{\left(\sin(x)\cdot\cos(x)\right)'}$$

$$\left(\sin(x)\cdot\cos(x)\right)' = \sin(x)\cdot\cos'(x) + \sin'(x)\cdot\cos(x)$$

by Leibniz formula

$$= -\sin^2 x + \cos^2 x$$

$$\tan(x^2)' = \tan'(x^2) \cdot (x^2)' =$$

Chain rule

$$= \tan'(x^2) \cdot 2x = \frac{1}{\cos^2(x^2)} \cdot 2x$$

$$\tan'(x) = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x) \cdot \sin'(x) - \cos'(x) \cdot \sin(x)}{\cos^2 x}$$

$$= \frac{\cos^2 x - (-\sin^2 x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

So we get

$$\left(\tan(x^2)+2\sin(x)\cdot\cos(x)\right)' =$$

$$\frac{1}{\cos^2(x^2)}\cdot 2x \quad + \quad 2\cdot\left(-\sin^2 x+\cos^2 x\right).$$

$\blacksquare$

New

Prop 2.21

5) Derivative of the inverse function

Let $f: (a,b) \to E$ be bijective continuous.

Let $x_0 \in (a,b)$ be s.t. f is differentiable at x_0 and $f'(x_0) \neq 0$. Then, f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(\underbrace{f^{-1}(y_0)}_{= x_0})}}$$

Example

$$f(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f^{-1}(y) = \arcsin(y)$$

$$\arcsin'(y) = \frac{1}{\sin'(\arcsin(y))} =$$

$$= \frac{1}{\cos(\arcsin(y))} =$$

$$\frac{1}{\cos(\arcsin(y))} =$$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} =$$

$$= \frac{1}{\sqrt{1-y^2}} \quad \blacksquare$$

Today:

- Exponential functions
- Left / Right derivatives
- higher derivatives
- class of C^k functions.

Exponential functions (Section 2.1.6)

Definition

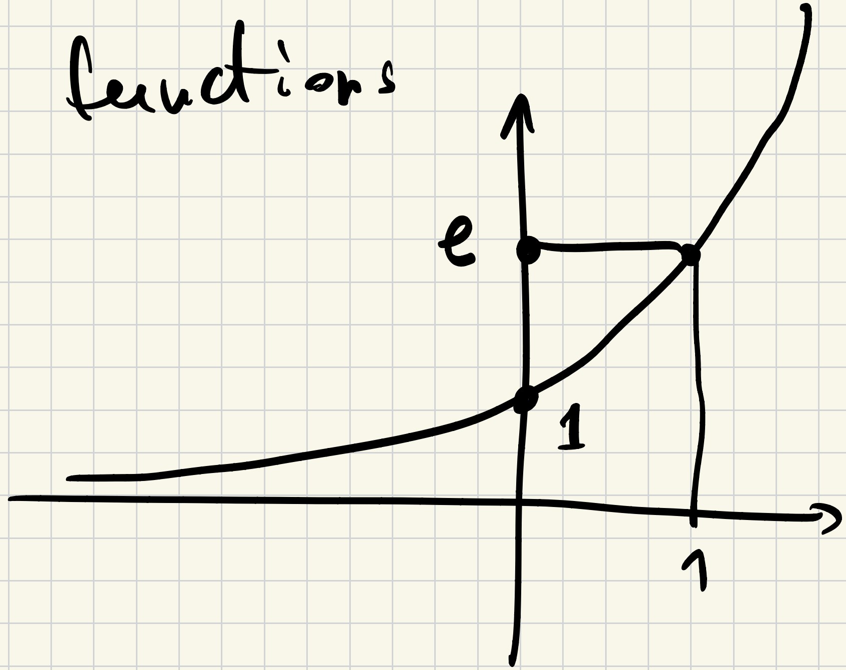
For $x \in \mathbb{R}$ we define

$$\boxed{e^x := 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}.}$$

If $x = 0$ then $\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{0}{1!} + \dots$

Remark: Series converges for every $x \in \mathbb{R}$

Properties of Exponential functions



Proposition

- $e^{x+y} = e^x \cdot e^y$
- $e^x > 0 \quad \forall x \in \mathbb{R}$
- $\lim_{x \to +\infty} e^x = +\infty$; $\lim_{x \to -\infty} e^x = 0$
- e^x is strictly increasing
- The image of e^x is $(0, +\infty)$.

Proposition $(e^x)' = e^x$

Proof: Want to show $\lim_{h \to 0} \frac{e^{(x_0+h)} - e^{x_0}}{h} = e^{x_0}$

$$\frac{e^{(x_0+h)} - e^{x_0}}{h} = \underbrace{e^{x_0}}_{\text{constant}} \cdot \left(\frac{e^h - 1}{h} \right)$$

It is enough to show that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

We will show that

$$\lim_{h \to 0} \left(\frac{e^h - 1}{h} - 1 \right) = 0$$

Notice if $|h| < 1$

$$0 \leq \left| \frac{e^h - 1}{h} - 1 \right| \leq \left| \frac{\sum_{k=0}^{\infty} \frac{h^k}{k!} - 1}{h} - 1 \right|$$

$$= \left| \sum_{k=1}^{\infty} \frac{h^{k-1}}{k!} - 1 \right| =$$

$$= \left| h \sum_{k=0}^{\infty} \frac{h^k}{k!} \right| \leq |h| \cdot \sum_{k=0}^{\infty} \frac{|h|^k}{k!} =$$

$$= |h| \cdot e^{|h|} \leq |h| \cdot e$$

$|h| \leq 1$ and e^x is increasing

So by Squeeze theorem we are done $\blacksquare$

Definitions 1) The natural logarithm

$\log(x) : \mathbb{R}_{>0} \to \mathbb{R}$ is the inverse to e^x.

2) For $a \in \mathbb{R}_{>0}$ we define

$$a^x := e^{x \cdot \underbrace{\log(a)}_{\text{constant}}}$$

$$\log_a(x) := \frac{\log(x)}{\underbrace{\log(a)}_{\text{constant}}}$$

constant

3) For $a \in \mathbb{R}$ we define $x^a := e^{a \cdot \log(x)}$

Example $\quad (\log(x))'$

Use formula for derivative of inverse function:

$(e^x)' = e^x$ so we get

$$\log'(x) = \frac{1}{e^{\log(x)}} = \frac{1}{x}.$$

__Example__ $f: \mathbb{R}_+ \to \mathbb{R}$ $h(x) = x^x$

$h(x) = g \circ f$ where

$g(y) = e^y$; $f(x) = x \cdot \log x$

$$\left(e^{x \cdot \log x} = \left(e^{\log x}\right)^x = x^x \right)$$

by Chain rule

$$h' = (g \circ f) = g'(f(x)) \cdot f'(x)$$

$$= e^{x \cdot \log(x)} \cdot (x \cdot \log(x))' =$$

$$= x^x \left(\underbrace{x'}_{=1} \cdot \log x + x \cdot \underbrace{\log(x)'}_{=\frac{1}{x}} \right) = x^x (\log x + 1)$$

One sided derivative (section 2.2.)

Definition Let $f: E \to \mathbb{R}$ and $x_0 \in E$

we say that left (right) derivative of f at x_0 exists if $\displaystyle\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$

$\left(\displaystyle\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \right)$ exists and finite.

<u>Definition</u> 1) Let $D = [a, b]$. We say $f : D \to \mathbb{R}$ is differentiable if it is

- differentiable for every $x \in (a, b)$
- Right differentiable at a
- Left differentiable at b

2) If $D = [a, +\infty)$, we say that $f : D \to \mathbb{R}$ is differentiable if it is

- differentiable for every $x \in (a, +\infty)$
- Right differentiable at a

3) If $D = (-\infty, b]$ we say that

$f : D \to \mathbb{R}$ is differentiable if it is

- differentiable for every $x \in (-\infty, b)$

- Left differentiable at b

Remark Function f is differentiable at $x_0 \in E$ if and only if it is left and right differentiable and

$$f'_{left}(x_0) = f'_{right}(x_0)$$

$$\lim_{x \to x_0^-} \boxed{\frac{f(x)-f(x_0)}{x-x_0}} \qquad \lim_{x \to x_0^+} \boxed{\frac{f(x)-f(x_0)}{x-x_0}}$$

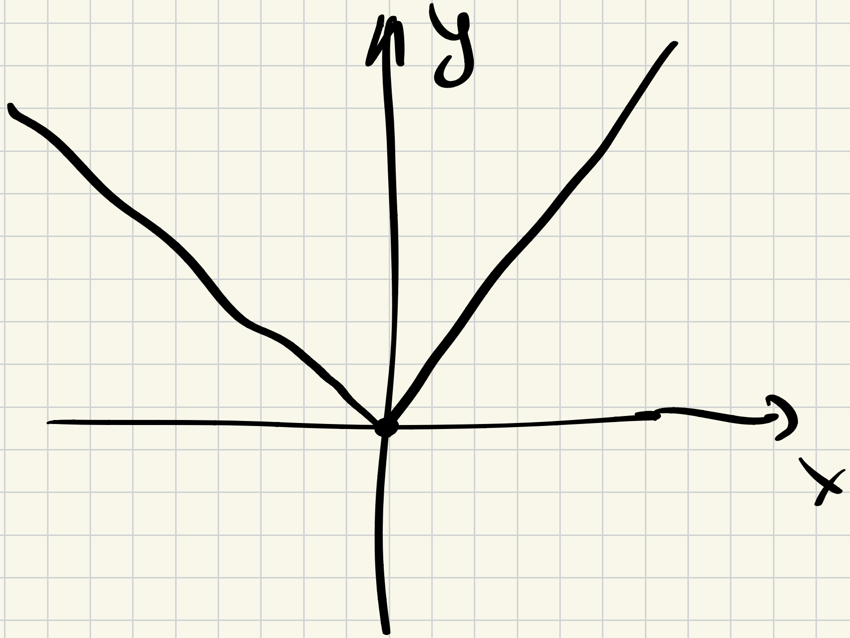
$$f'(x_0) = \lim_{x \to x_0} \boxed{\frac{f(x) - f(x_0)}{x - x_0}}$$

If $\displaystyle\lim_{x \to x_0^-} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) =$

$\displaystyle= \lim_{x \to x_0^+} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) = \ell \implies$

two sided limit $= \ell$ so $f'(x_0) = \ell$

Example $f(x) = |x|$



$f(x)$ it is not differentiable at 0.

But it is both left and right differentiable:

$$\lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

Similarly:

$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

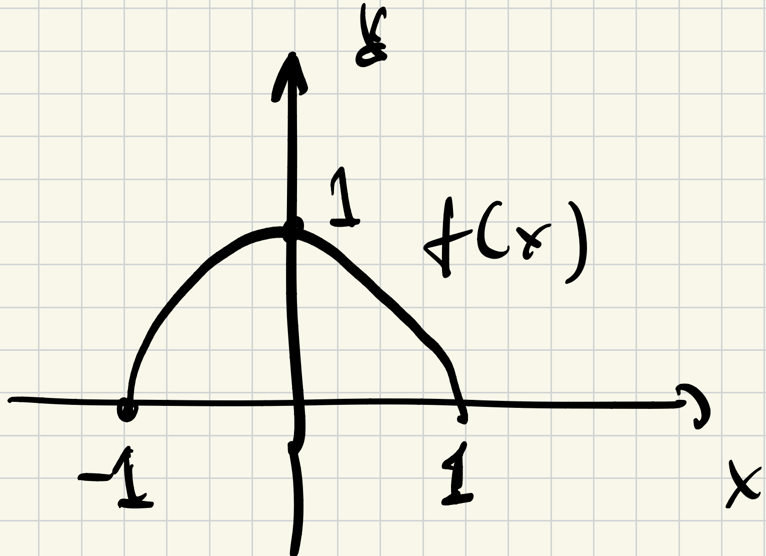
So left derivative is -1

right derivative is $+1$

$\blacksquare$

Example Consider $f(x) = \sqrt{1-x^2}$

on $[-1, 1]$



Is $f(x)$ differentible?

- for $x \in (-1, 1)$ $f(x)$ is differentiable since it composition of $\sqrt{x}$ and $1-x^2$ which are differentible

Check for Right differentiability at $x_0 = -1$!

$$\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1^+} \frac{\sqrt{1-x^2} - 0}{x+1}$$

$$= \lim_{x \to -1^+} \frac{\sqrt{1-x} \cdot \sqrt{1+x}}{1+x} = \lim_{x \to -1^+} \frac{\sqrt{1-x}}{\sqrt{1+x}} =$$

$\sqrt{2} > 0$

$\to 0$

> 0

$$= +\infty$$

So the right derivative of $f(x) = \sqrt{1-x^2}$ does not exist at $x_0 = -1$.

Similarly, left derivative does not exists at $x_0 = 1$.

$\Rightarrow$ $f(x)$ is not differentiable on $[-1, 1]$. ■

Higher derivatives and C^k functions.

Definition 2.42. Let $f: E \to \mathbb{R}$, then

the second derivative $f''(x_0)$ (or $f^{(2)}(x_0)$)

of f at x_0 is given by

$$f''(x_0) = \left(f'\right)'(x_0)$$

<u>That is</u>

$$\boxed{f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}}$$

- If we defined the n-th derivative $f^{(n)}(x)$ then the $(n+1)$-st derivative $f^{(n+1)}$ of f is given by

$$f^{(n+1)}(x) := \left(f^{(n)}\right)'(x).$$

Example . $f(x) = e^x$

$$f'(x) = e^x$$

$$f''(x) = (e^x)' = e^x$$

$$\Rightarrow \quad f^{(n)}(x) = e^x \quad .$$

Example $f(x) = x^4$

$$f'(x) = 4 \cdot x^3$$

$$f''(x) = (4 \cdot x^3)' = 4 \cdot (3x^2) = 12x^2$$

$$f'''(x) = (f''(x))' = (12x^2)' = 12 \cdot 2x = 24x$$

$$f''''(x) = (24x)' = 24$$

$$f^{(5)}(x) = 0 \Rightarrow f^{(n)} = 0 \text{ for } n \geq 5$$

Definition Let $D \subseteq \mathbb{R}$ and let

$f: D \to \mathbb{R}$ be a function.

1) If f is differentiable on D and f' is continuous on D we say that f is continuously differentiable or C^1-function

$$C^1(D) = \left\{ f \in C^0(D) \;\middle|\; f \text{ is continuously differentiable} \right\}$$

$$= \left\{ f \in C^0(D) \;\middle|\; f \text{ is differentiable and } f' \in C^0(D) \right\}$$

(ii) In general, for $n \in \mathbb{N}$ we say that **f is of class C^n** if

All derivatives $f^{(k)}$ of order $k \leq n$ exist and continuous.

<u>Remark</u> since $f^{(n)}$ exists

$f^{(n-1)}$ is differentiable it should also be continuous

...

$$C^n(D) = \left\{ f : D \to \mathbb{R} \;\middle|\; f, f', f'', \dots f^{(n)} \text{ exist and continuous} \right\} =$$

$$= \left\{ f \in C^{n-1}(D) \;\middle|\; f^{(n-1)} \in C^1(D) \right\}$$

Definition We define $C^\infty(D) = \bigcap_{k=1}^{\infty} C^k(D)$

$C^\infty(D)$ = class of infinitely differentiable functions.